

Minimum Moment Steiner Trees*

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Abstract

For a rectilinear Steiner tree T with a root, define its k -th moment

$$M_k(T) = \int_T (d_T(u))^k du,$$

where the integration is over all edges of T , $d_T(u)$ is the length of the unique path in T from the root to u , and du is the incremental edge length. Given a set of points P in the plane, a k -th moment Steiner Minimum Tree (k -SMT) is a rectilinear Steiner tree that has the minimum k -th moment among all rectilinear Steiner trees for P , with the origin as the root. The definition is a natural extension of the traditional Steiner minimum tree, and motivated by application in VLSI routing. In this paper properties of the k -SMT are studied and approximation algorithms are presented.

1 Introduction

We are given a set of points $P = \{p_0, p_1, \dots, p_n\}$ in the plane, where $p_0 = (0, 0)$, $p_i = (x_i, y_i)$, and x_i, y_i are integers. An *edge* is a horizontal or vertical line segment. A *rectilinear tree* is a connected acyclic collection of edges, which intersect only at the endpoints, which we call nodes. We use $\|p_i\| = |x_i| + |y_i|$ to represent the rectilinear distance between p_0 and p_i .

A *rectilinear Steiner tree* [2, 4] for P is a rectilinear tree such that each point $p_i \in P$ is a node in the tree. A *rectilinear Steiner Minimum Tree* (SMT) for P is

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a rectilinear Steiner tree for P that has the shortest total edge length. A *rectilinear Steiner arborescence* [5, 8] for P is a rectilinear Steiner tree for P such that for each $p_i \in P$, the length of the unique path in the tree from root p_0 to p_i is $\|p_i\|$. A *rectilinear Steiner Minimum Arborescence* (SMA) for P is a rectilinear Steiner arborescence for P that has the shortest total edge length.

Since the whole paper is about rectilinear Steiner trees, we omit the word “rectilinear” in the rest of the paper.

Definition 1.1 For a Steiner tree T with a root, define its k -th moment:

$$M_k(T) = \int_T (d_T(u))^k du,$$

where the integration is over all edges of T , $d_T(u)$ is the length of the unique path in T from the root to u , and du is the incremental edge length. Given a set of points P in the plane, a k -th moment Steiner Minimum Tree (k -SMT) is a Steiner tree that has the minimum k -th moment among all Steiner trees for P , with the origin as the root.

Clearly, 0-SMT is the traditional SMT. Figure 1 shows the SMT, 1-SMT and SMA for a set of points. $M_0(T_1) = 14$, $M_0(T_2) = 15$, $M_0(T_3) = 16$, $M_1(T_1) = 98$, $M_1(T_2) = 97.5$, and $M_1(T_3) = 98$.

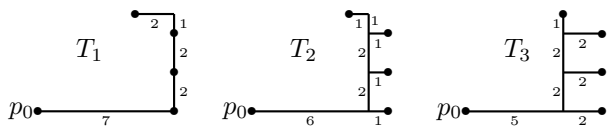


Figure 1: T_1 is an SMT, T_2 is a 1-SMT, and T_3 is an SMA.

The new formulation is motivated by the advances in Very Large Scale Integration (VLSI). As the feature size of integrated circuits reduces, the wire delay

estimation based the total wire length is no longer accurate. In 1983, Rubinstein, Penfield and Horowitz [7] proposed to use the first moment of the routing tree, also known as the Elmore delay, to estimate the wire delay. In 1993, Cong, Leung and Zhou [1] defined quadratic Steiner tree, which is our 1-SMT. However, there has been no study on the properties of k -SMT for any $k \geq 1$, nor any approximation algorithm.

This paper contains the following results:

- For any constant $k \geq 1$, k -SMT is NP-hard, but not known in NP.
- For any constant $k \geq 1$, the k -th moment of an SMT can be a factor of $\Omega(n^{k/2})$ worse than that of a k -SMT, and the k -th moment of an SMA can be a factor of $\Omega(\log n)$ worse than that of a k -SMT.
- Given P , it is obvious any 0-SMT for P is an SMT. Any ∞ -SMT for P is a Steiner arborescence.
- An $(8/3)c$ -approximation algorithm for the 1-SMT problem, where $c \leq 1 + \ln 3/2$ is the performance for the approximation of Steiner trees in graphs.
- An $f(k)$ -approximation algorithm for the k -SMT problem, where $f(k)$ is a function of k but independent of n .

2 Properties

Theorem 2.1 *The k -SMT problem is NP-hard for any constant $k \geq 0$.*

Proof. For $k = 0$, the problem is the same as the SMT problem, a well known NP-Complete problem [2]. For $k \geq 1$, we use a reduction from SMT. Given an instance P for the SMT problem, let the leftmost point of P be $p_i = (x_i, y_i)$. Let the minimum square that contains P be size $L \times L$. Define the new root $p_0 = (x_i - W, y_i)$, where $W = 2^k L^4$, see Figure 2. Now we show for any Steiner trees T and T' of P , $M_0(T) < M_0(T')$ if and only if $M_k(T \cup \{(p_0, p_i)\}) < M_k(T' \cup \{(p_0, p_i)\})$.

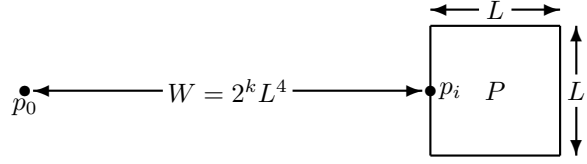


Figure 2: Construction used for reduction.

If $M_0(T) < M_0(T')$, then

$$\begin{aligned}
 & M_k(T \cup \{(p_0, p_i)\}) - M_k(T' \cup \{(p_0, p_i)\}) \\
 & < M_0(T)(W + L^2)^k - M_0(T')W^k \\
 & \leq (M_0(T') - 1)(W + L^2)^k - M_0(T')W^k \\
 & = M_0(T')((W + L^2)^k - W^k) - (W + L^2)^k \\
 & < L^2(2^k L^2 W^{k-1}) - (W + L^2)^k \\
 & = W^k - (W + L^2)^k \\
 & < 0.
 \end{aligned}$$

□

It is not known if the k -SMT problem is in NP, since the k -SMT problem does not have the Hanan property as shown below.

Definition 2.1 *The Hanan grid of a set of points P is the grid $G(P)$ that includes a vertical line and a horizontal line through every point $p_i \in P$. A problem has Hanan property if there exists an optimal solution that uses only edges of $G(P)$.*

Hanan property is useful since it reduces the solution space and makes the problem in NP. Both the SMT problem and the SMA problem have Hanan property [3, 5].

Theorem 2.2 *The k -SMT problem does not have Hanan property for any $k \geq 1$.*

Proof. In Figure 1, the 1-SMT does not use edges of $G(P)$. Similar examples can be constructed for general k . In general, the solutions are not rational. □

Theorem 2.3 *For any constant $k \geq 1$, let $T_{SMT}(P)$ be an SMT for P and $T_{k-SMT}(P)$ be a k -SMT for P .*

Then

$$\max_P \frac{M_k(T_{SMT}(P))}{M_k(T_{k-SMT}(P))} = \Omega(n^{k/2}).$$

Proof. In Figure 3, there are n points placed in $\Theta(\sqrt{n})$ rows, with $\Theta(\sqrt{n})$ points on each odd row, and 1 point on each even row. It was shown the length of T_{SMA} for the right half is size Then it is easy to see

$$M_k(T_{SMT}(P)) = \Theta(n^{k+1}),$$

$$M_k(T_{SMA}(P)) = \Theta(n^{k/2+1}).$$

□

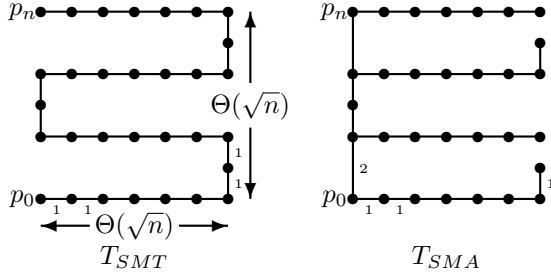


Figure 3: SMT is not a good approximation for k -SMT.

Theorem 2.4 For any constant $k \geq 1$, let $T_{SMA}(P)$ be an SMA for P and $T_{k-SMT}(P)$ be a k -SMT for P . Then

$$\max_P \frac{M_k(T_{SMA}(P))}{M_k(T_{k-SMT}(P))} = \Omega(\log n).$$

Proof. In Figure 4, there are n points equally spaced on the diagonal line from p_1 to p_n . It was shown in [5] that the total edge length for the right half of T_{SMA} is $\Theta(n \log n)$. Therefore, it is easy to verify

$$M_k(T_{SMA}) = \Omega(n^{k+1} \log n),$$

$$M_k(T_{SMT}) = \Theta(n^{k+1}).$$

□

Theorem 2.5 For any P , $\lim_{k \rightarrow \infty} k$ -SMT(P) is a Steiner arborescence.

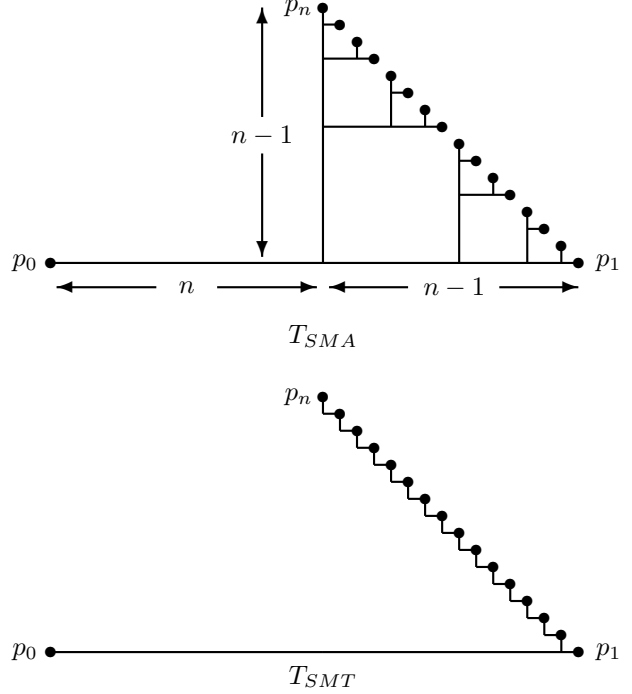


Figure 4: SMA is not a good approximation for k -SMT.

Proof. Let (s, t) be any non-monotone path of length $\epsilon > 0$ without branch in a k -SMT. By non-monotone, we mean the path from the root to t goes through s , and $\|s\| > \|t\|$. We show there exists a $K = K(\epsilon)$ such that when $k > K$, such non-monotone path can not exist in any k -SMT.

The k -th moment of path (s, t) is

$$\begin{aligned} \int_{d_T(s)}^{d_T(t)} u^k du &\geq \int_{\|s\|}^{\|s\|+\epsilon} u^k du \\ &> \int_{\|t\|}^{\|t\|+\epsilon} u^k du \\ &= \frac{1}{k+1} ((\|t\| + \epsilon)^{k+1} - \|t\|^{k+1}) \\ &> \epsilon \cdot \|t\|^k. \end{aligned}$$

On the other hand, a direct path from the root to t will have k -th moment $\|t\|^{k+1}/(k+1)$. Therefore when $k \geq K = \|t\|/\epsilon - 1$, we have $\epsilon \geq \|t\|/(k+1)$ and

$$\epsilon \cdot \|t\|^k \geq \frac{1}{k+1} \|t\|^{k+1}.$$

That means a direct path (p_0, t) will have less k -th moment than the (s, t) path. Therefore such (s, t) path cannot exist in any k -SMT. \square

Even though an ∞ -SMT must be a Steiner arborescence, an ∞ -SMT is not necessarily an SMA, see Figure 5.

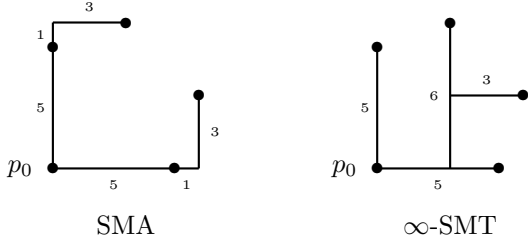


Figure 5: An SMA is not necessarily an ∞ -SMT.

3 Approximation Algorithm

For any edge e , $M_k(e)$ depends on the path from root to e . Therefore, any change in one part of the tree affects the k -th moment in other parts of the tree. This makes it difficult to apply most approximation techniques on the k -SMT problem for any $k \geq 1$. In the following, we introduce another cost function that is closely related to M_k , but is independent of the rest of the tree.

Definition 3.1 For any Steiner tree T , define the k -th direct cost

$$D_k(T) = \int_T \|u\|^k du,$$

where $\|u\| = |x_u| + |y_u|$ is the rectilinear distance from the origin to u . For a given set of points P and the origin as the root, a rectilinear k -th Direct Cost Steiner Minimum Tree (k -DMT) is a rectilinear Steiner tree T such that $D_k(T)$ is the minimum among all rectilinear Steiner trees for P .

Lemma 3.1 For any P , let $T_{k-SMT}(P)$ be a k -SMT for P and T_{k-DMT} be a k -DMT for P . Then

$$M_k(T_{k-SMT}(P)) \geq D_k(T_{k-DMT}(P)).$$

Proof. $M_k(T_{k-SMT}(P)) \geq D_k(T_{k-SMT}(P)) \geq D_k(T_{k-DMT}(P)).$ \square

Lemma 3.2 The k -DMT problem has Hanan property.

Proof. We prove that for any set of points P , there exists a k -DMT that uses only edges of $G(P)$. The proof is an adaptation of Hanan's proof for rectilinear Steiner tree problem [3].

Suppose we have a k -DMT T for P , and a grid $G(P)$. Let I be the set of intersection points on $G(P)$. Let Q be the set of nodes of T that are not in I . Points in Q are either Steiner points with degree 3 or 4, or corner points with degree 2. Let I -point denote a point in I and Q -point a point in Q . In the following we will show all Q -points of T can be removed to obtain a new tree T' such that $D_k(T) = D_k(T')$.

For any Q -point s , let $N_I(s)$ be the number neighbors of s that are in I . Let $d(s)$ be the degree of point s . Table 1 shows the cases.

For Case 1, it is easy to see $s \in I$.

For Case 2, if s is a Steiner point and $d(s) = 4$, s can be seen as 2 Steiner points s_1 and s_2 , and $d(s_1) = d(s_2) = 3$. Therefore we only need to consider $d(s) = 3$.

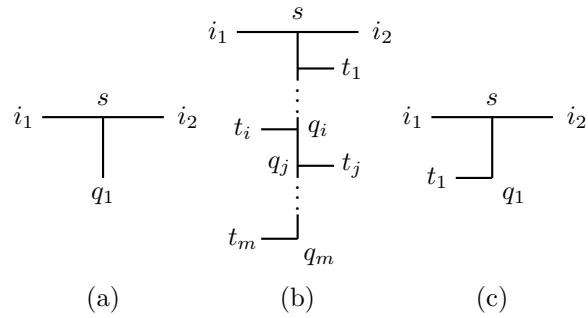


Figure 6: Case 2, where $s \in Q$, $d(s) = 3$, $N_I(s) = 2$.

Let the 2 neighboring I -points of s be i_1 and i_2 , and the 1 neighboring Q -point be q_1 . Without loss of generality, we can assume that i_1 is on the left of s , i_2 is on the right of s , and q_1 is below s , see Figure 6(a). If i_1 is on the left of s and i_2 is below s , then clearly $s \in I$. Furthermore, assume the X -coordinate

Table 1: Cases in proving Hanan property for k -DMT.

		$N_I(s)$				
		0	1	2	3	4
$d(s)$	2	Case 4	Case 3	Case 1	/	/
	3	Case 4	Case 4	Case 2	Case 1	/
	4	Case 4	Case 4	Case 2	Case 1	Case 1

of s is greater than 0. Otherwise, reflect the graph about Y -axis.

If we follow the edge sq_1 downward, we might find some edges $q_i t_i$ going left, and some edges $q_j t_j$ going right, and q_i 's and q_j 's may overlap, see Figure 6(b). Note that $q_1 t_1$ must go right, since otherwise we will be in Figure 6(c), and by moving edge sq_1 to left we can reduce the cost. Let q_m be the end of the sequence of edges from s that are on the same vertical line. Edge $q_m t_m$ may go either left or right. Let $L = \{q_i \mid q_i t_i \text{ goes to left}\}$, $R = \{q_j \mid q_j t_j \text{ goes to right}\}$, $l = |L|$ and $r = |R|$.

Move all vertical edges from s to q_m to left by ϵ , and correspondingly, increase the length of edges going to right and decreasing the length of edges going to left. Call the new tree T' .

First consider the case $k = 1$.

$$\begin{aligned}
& D_1(T) - D_1(T') \\
&= (\|sq_m\|) \cdot \epsilon + \sum_{q_i \in L} \int_{\|q_i\| - \epsilon}^{\|q_i\|} u du - \sum_{q_j \in R} \int_{\|q_j\| - \epsilon}^{\|q_j\|} u du \\
&= (\|sq_m\|) \cdot \epsilon + \sum_{q_i \in L} \frac{2\|q_i\|\epsilon - \epsilon^2}{2} - \sum_{q_j \in R} \frac{2\|q_j\|\epsilon - \epsilon^2}{2} \\
&= \left(\|sq_m\| + \sum_{q_i \in L} \|q_i\| - \sum_{q_j \in R} \|q_j\| + \frac{(r-l)\epsilon}{2} \right) \epsilon.
\end{aligned}$$

For every i , if $q_i t_i$ goes to left, $q_{i+1} t_{i+1}$ must go to right, otherwise moving edge $q_i q_{i+1}$ to left will reduce the cost, which is a contradiction to the fact that T is a 1-DMT. Therefore, $r - l \geq 0$ except for one case, which is in Figure 6(c) (but clearly in this case moving edge sq_1 to left reduces the cost).

If $(\|sq_m\| + \sum_{q_i \in L} \|q_i\| - \sum_{q_j \in R} \|q_j\|) > 0$, then for any $\epsilon > 0$, $D_1(T) - D_1(T') > 0$, contradic-

tion to the fact that T is a 1-DMT. If $(\|sq_m\| + \sum_{q_i \in L} \|q_i\| - \sum_{q_j \in R} \|q_j\|) < 0$, then for any $\epsilon < 0$, $D_1(T) - D_1(T') > 0$, again a contradiction.

If $(\|sq_m\| + \sum_{q_i \in L} \|q_i\| - \sum_{q_j \in R} \|q_j\|) = 0$, and $r - l > 0$, then $D_1(T) - D_1(T') > 0$ for any $\epsilon \neq 0$.

Finally, if $(\|sq_m\| + \sum_{q_i \in L} \|q_i\| - \sum_{q_j \in R} \|q_j\|) = 0$, and $r - l = 0$, then $D_1(T) - D_1(T') = 0$ for any ϵ . When this happens, we can move all edges from s to q_m to the left or right, to either make s an I-point, or remove another Q-point.

For $k \geq 2$, we prove that T is not a k -DMT by contradiction.

For the time being, assume q_m is also in the first quadrant. Let

$$\begin{aligned}
\Delta(\epsilon) &= D_k(T) - D_k(T') \\
&= \int_{\|q_m\|}^{\|s\|} u^k du - \int_{\|q_m\| - \epsilon}^{\|s\| - \epsilon} u^k du \\
&\quad + \sum_{q_i \in L} \int_{\|q_i\| - \epsilon}^{\|q_i\|} u^k du - \sum_{q_j \in R} \int_{\|q_j\| - \epsilon}^{\|q_j\|} u^k du.
\end{aligned}$$

Since T_k is a k -SMT, $\Delta(\epsilon)$ reaches local maximum at $\epsilon = 0$. Therefore

$$\begin{aligned}
\left. \frac{d\Delta}{d\epsilon} \right|_{\epsilon=0} &= \|s\|^k + \sum_{q_i \in L} \|q_i\|^k - \sum_{q_j \in R} \|q_j\|^k - \|q_m\|^k \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\left. \frac{d^2\Delta}{d\epsilon^2} \right|_{\epsilon=0} &= k \left(\sum_{q_j \in R} \|q_j\|^{k-1} + \|q_m\|^{k-1} \right) \\
&\quad - k \left(\|s\|^{k-1} + \sum_{q_i \in L} \|q_i\|^{k-1} \right) \\
&\leq 0.
\end{aligned}$$

Let $s = q_0$ and $L' = L \cup \{q_0\}$. Let $q'_m = q_m$ and $R' = R \cup \{q'_m\}$. Since

$$\sum_{q_i \in L'} \|q_i\|^k = \sum_{q_j \in R'} \|q_j\|^k,$$

and the property that $s \geq q_1 \geq \dots \geq q_m$, it can be shown that there exist lr positive real numbers $\alpha_{i,j}$ such that for each $q_j \in R'$,

$$\sum_i \alpha_{i,j} = 1,$$

and for each $q_i \in L'$,

$$\|q_i\|^k = \sum_{q_j \in R'} \alpha_{i,j} \|q_j\|^k,$$

where $\alpha_{i,j} = 0$ for $\|q_j\| > \|q_i\|$. Such $\alpha_{i,j}$'s must exist, since otherwise, we would be able to move edges between s and q_j , for some $q_j \in R'$, to the right to reduce the k -th directed cost of T . Then, we have

$$\begin{aligned} \|q_i\|^{k-1} &= \sum_{q_j \in R'} \alpha_{i,j} \|q_j\|^k / \|q_i\| \\ &< \sum_{q_j \in R'} \alpha_{i,j} \|q_j\|^{k-1}. \end{aligned}$$

Therefore,

$$\sum_{q_i \in L'} \|q_i\|^{k-1} < \sum_{q_j \in R'} \|q_j\|^{k-1},$$

which contradicts with the fact that $(d^2 \Delta / d\epsilon^2) |_{\epsilon=0} \leq 0$.

If q_m is in the fourth quadrant, then either $d(q_m) = 3$ or $q_m t_m$ goes to right, otherwise we can flip the q_m corner from shape “]” to shape “[” to reduce the direct cost. Thus, we can divide the subgraph into two parts, one in the first quadrant and the other in the fourth quadrant, then we get similar cases, which will also lead to a contradiction.

Case 3. Now consider the case that s is a corner point $d(s) = 2$ and $N_I(s) = 1$. This case can be viewed as a special case of Figure 6(a), by removing point i_1 and edge $s i_1$, or removing point i_2 and edge $s i_2$. Then a similar argument shows s can be removed from Q .

Case 4. The following argument proves that Q is empty after removing all s that are in Cases 1, 2 and

3 from Q . Assume the contrary, that is, if Q is not empty, any point s remaining in Q is either $d(s) \geq 3$ and $N_I(s) \leq 1$, or $d(s) = 2$ and $N_I(s) = 0$. It implies that for any s in Q , there are at least 2 edges, each connects s with another Q -point. Suppose there are m such points remaining in Q . It's easy to see that for the subgraph with those m points, there are at least m edges in this subgraph. This implies that there exists a cycle in this subgraph. Obviously it cannot happen. Therefore, Q is empty. \square

Given a graph $G = (V, E)$ with weight $w : E \rightarrow [0, +\infty)$ and a subset of vertices $S \subset V$, a Steiner tree T for S is a connected subgraph that contains all the vertices in S and $\sum_{e \in T} w(e)$ is minimum. Finding Steiner tree for graphs is NP-Complete. The best approximation algorithm is a factor of $1 + \ln 3/2$ from the optimal [6].

For each edge e of Hanan grid $G(P)$, assign weight $w(e) = D_k(e)$. Lemma 3.2 says that in order to find a k -DMT, we can find an SMT for graph $G(P)$ with weight w .

Our approximation algorithm is given in Figure 7. At step 7 of the algorithm, the factor 3 is from careful analysis for $k = 1$ in order to give the best performance bound.

Let v_1, v_2, \dots, v_m be the nodes to which the algorithm adds direct paths from the root p_0 . Let R_1, \dots, R_{m+1} be the depth-first traversal of T_d , where R_i is between v_{i-1} and v_i . Let L_1, \dots, L_m be the paths added, where L_i is from p_0 to v_i . Figure 8 shows one part of the graph generated by the above algorithm. In the figure, R_i is the traversal path from v_{i-1} to v_i . There may be some nodes in the path. When L_i is added, the length of the path from p_0 through v_{i-1} to v_i is $3\|v_i\|$, or $(|L_{i-1}| + |R_i|) = 3\|v_i\| = 3|L_i|$.

Lemma 3.3 *In step 10 of the algorithm when L_i is added, $M_1(L_{i-1}) + M_1(R_i) + M_1(L_i) \leq \frac{8}{3} D_1(R_i)$.*

Proof. Paths L_{i-1} , L_i and R_i form a cycle of length

$$|L_{i-1}| + |R_i| + |L_i| = 4|L_i|.$$

Therefore,

$$M_1(L_{i-1}) + M_1(R_i) + M_1(L_i)$$

Algorithm 1 Approximation algorithm for 1-SMT.

Input: $P = \{p_1, \dots, p_n\}$ is a set of points and root $p_0 = (0, 0)$.

Output: T is an approximation 1-SMT.

- 1: Construct $G(P)$ and assign weight $w(e) = D_1(e)$ for each edge e ;
- 2: Find Steiner minimum tree T_d for P in graph $G(P)$;
- 3: Traverse edges of T_d in depth-first order; Let e_1, e_2, \dots, e_N be the sequence of edges of T_d in the traversal, where each edge appears twice;
- 4: Let $i \leftarrow 1, T \leftarrow \emptyset$;
- 5: **while** $i \leq N$ **do begin**
- 6: Consider edge $e_i = (u_i, u_{i+1})$;
- 7: **if** $d_T(u_i) + \|(u_i, u_{i+1})\| < 3\|u_{i+1}\|$ **then**
- 8: $T \leftarrow T \cup \{e_i\}, i \leftarrow i + 1$;
- 9: **else** find point $q \in e_i$ such that $d_T(u_i) + \|(u_i, q)\| = 3\|q\|$;
- 10: $T \leftarrow T \cup \{(p_0, q), (q, u_i)\}, u_i \leftarrow q$;
- 11: **end;**
- 12: **for** each $p_i \in P$ **do**
- 13: Mark the edges of T on the shortest path from p_i to p_0 ;
- 14: Delete all edges that are not marked.

Figure 7: Approximation algorithm for 1-SMT.

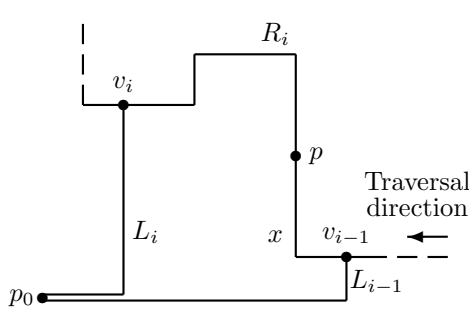


Figure 8: Part of depth-first traversal R , with direct paths added.

$$= 2 \cdot \int_0^{2|L_i|} u du = 4|L_i|^2.$$

On the other hand, consider any point p on R_i . Let the length of path from v_{i-1} to p be x . Then from the algorithm, the length of path (p_0, p) through v_{i-1} is at most $3\|p\|$. In addition, $\|p\| \geq |L_{i-1}| - x$. Therefore,

$$\begin{aligned} D_1(R_i) &= \int_{u \in R_i} \|u\| du \\ &\geq \int_0^{|R_i|} \max \left\{ \frac{|L_{i-1}| + x}{3}, |L_{i-1}| - x \right\} dx \end{aligned}$$

$$\begin{aligned} &\geq \int_0^{\frac{1}{2}|L_{i-1}|} (|L_{i-1}| - x) dx \\ &\quad + \int_{\frac{1}{2}|L_{i-1}|}^{3|L_{i-1}| - |L_{i-1}|} \frac{|L_{i-1}| + x}{3} dx \\ &= \frac{3}{2} |L_{i-1}|^2. \end{aligned}$$

□

Theorem 3.4 Given a set of points, Algorithm 1 finds a Steiner tree T in polynomial time such that $M_1(T) \leq \frac{8}{3}c \cdot M_1(T_{1-SMT}(P))$, where c is the factor for best approximation algorithm for Steiner trees in graphs, and $T_{1-SMT}(P)$ is a 1-SMT for P .

Proof. The time complexity is bounded by the time to find Steiner trees for graphs in step 2 of the algorithm. All other steps are low order polynomial.

Suppose we have found T_d which is a c -approximation 1-DMT. Let v_1, v_2, \dots, v_m be the nodes to which the algorithm adds direct paths from the root p_0 . Let R_1, \dots, R_{m+1} be the depth-first traversal of T_d , where R_i is between v_{i-1} and v_i . Let L_1, \dots, L_m

be the paths added, where L_i is from p_0 to v_i . For any point $u \in T_d$, there are 2 corresponding copies u', u'' in $R = \{R_1, \dots, R_{m+1}\}$, because the traversal goes through u twice. Suppose u' appears in R_j and u'' appears in R_k , and $d_T(u')$ is the length from the root p_0 to u through R_j and $d_T(u'')$ the length through R_k . Because in step 14, we keep the shortest path from p_0 to u ,

$$d_T(u) = \min\{d_T(u'), d_T(u'')\} \leq \frac{d_T(u') + d_T(u'')}{2}.$$

Therefore,

$$\begin{aligned} M_1(T) &\leq \sum_{i=1}^m M_1(L_i) + \int_{u \in T_d} \frac{d_T(u') + d_T(u'')}{2} du \\ &\leq \sum_{i=1}^m M_1(L_i) + \frac{1}{2} \sum_{i=1}^{m+1} M_1(R_i) \\ &= \frac{1}{2} \sum_{i=1}^{m+1} (M_1(L_{i-1}) + M_1(R_i) + M_1(L_i)). \end{aligned}$$

Here, we let $M_1(L_0) = M_1(L_{m+1}) = 0$ to satisfy the boundary condition. From Lemma 3.3,

$$M_1(T) \leq \frac{1}{2} \sum_{i=1}^{m+1} \frac{8}{3} D_1(R_i) \leq \frac{4}{3} \cdot 2D_1(T_d) = \frac{8}{3} D_1(T_d).$$

On the other hand, let T' be optimal Steiner tree for $G(P)$, then from Lemma 3.1,

$$D_1(T_d) \leq c \cdot D_1(T') \leq c \cdot M_1(T_{1-SMT}).$$

Therefore, $M_1(T) \leq \frac{8}{3}c \cdot M_1(T_{1-SMT})$. \square

Because the Hanan property is true for k -DMT, we have

Theorem 3.5 *Algorithm 1 works for any k , except that the constant 3 in steps 7 and 9 has to be changed to $\beta_k = 2^{(k+1)/k} - 1$. The performance is bounded by*

$$f(k) = \frac{(\beta_k + 1)^{k+1}}{\beta_k 2^k} c,$$

where $c \leq 1 + \ln 3/2$ is the performance for the approximation of Steiner trees in graphs.

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